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# NONSTANDARD FINITE DIFFERENCE SCHEME FOR HELMHOLTZ WAVE EQUATION

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## Abstract

*The wave equation is a well known partial differential equation (PDE) very often used to describe phenomena observed in many areas of physical sciences including mechanics, electromagnetics, quantum mechanics, relativistic gravity and acoustics. When transformed into frequency domain it forms Helmholtz equation. However, the numerical solution of these partial differential equations by standard discretisation schemas leads to the inverse problem which appears to be ill-conditioned for the case of Helmholtz equation.*

*The papers presents the basics of nonstandard scheme and its application for the case of the modelling the frequency response of piezoelectric circular disc. The results show its superior properties in solving discrete linear systems originated from transforming continues Helmholtz partial differential equation into discrete domain. The included example represents unique illustration of results obtained with standard and nonstandard schemas.*

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## Introduction

Mathematical models of many dynamical systems very often leads to the partial differential equations (PDE). A variety of methods have been constructed to calculate the solution for arbitrary PDE with arbitrary initial and boundary conditions. One of the first method was finite difference method (FDM) that approximates the solution using finite difference equations. The method require to divide analyzed space into very refined mesh to obtain satisfactory results. The concept of solution based on dividing the domain into subdomain was further developed by introducing variational methods from the calculus of variations to solve the problem by minimizing an associated error function. This numerical technique is now known as finite element method (FEM). Moreover in the last years the divergence theorem was applied in a process of solution for some of equations that could be transformed into integral form. The solution based on transforming volume integrals into surface integral is essential in the method called finite volume method (FVM).

In engineering practice when the system is linear the partial difference equations having time dependence could be transformed into frequency domain what reduces the dimensionality but introduces calculations on complex numbers. In all cases the problem of accuracy and its dependence on the size of the mesh is still chalanging despite the fact that high power computers are available at your fingertips.

## Theory

The standard approach of numerical solution of PDE applies the approximation of derivatives with following scheme:

$$\frac{d\psi}{dx} \approx \frac{\psi_{i+1} - \psi_i}{h} \tag{1}$$

where *h* is typically called sampling distance. The accuracy of results obtained by that approach is heavily dependent on *h* with theoretically exact solution obtained with limiting case of *h*=0, what is impossible to implement numerically. The error coming from using nonzero values of sampling space obviously could be identified when Taylor expansion of a function is considered.

As a remedy for minimizing notorious error in an approximation of derivatives Mickens [1] proposed different approach with discretization of the derivative described by:

$$\frac{d\psi}{dx} \approx \frac{\psi_{i+1} - \psi_i}{\phi(h)} \tag{2}$$

where $\phi(h)$ has different form depending on the PDE. He gives examples of several PDEs that could be exactly solved with discrete approach when denominator function has the form i.e. $\sin(h), e^h - 1, 1 - e^{-h}, \sinh(h), \ln(1+h), (1 - e^{\tau h})/\tau$. He defines also nonstandard difference rules as proposed by his theory that could be formulated as follows: 1) discrete derivative has more complicated denominator, 2) nonlinear term should be modeled nonlocally $y^2 => y_{i+1}y_i$, 3) the orders of the discrete derivatives must equal to the orders of the corresponding derivatives of differential equations, 4) special solutions of the differential equations should be a solution of the finite difference equation, 5) the finite difference equations should not have solutions that do not correspond exactly to solutions of the differential equations. However it is not straightforward how to attribute the denominator function for an arbitrary PDE. But he states that applying the rules mentioned gives consistency, stability and convergence when solving PDE by numerical algorithms.

The application of nonstandard scheme proposed by Mickens theory to Helmholtz equation leads to the lemma that can be formulated as follows: when solving one-dimensional Helmholtz PDE $\Delta\psi + k^2\psi = 0$ with a discrete scheme

$$\frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{h^2} + k^2\psi_i = 0 \tag{3}$$

the exact solution is obtained when replacing: $h \rightarrow h[\sin(kh/2)/(kh/2)]$ in differential equation (in second derivative) and $h \rightarrow h[\sin(kh)/kh]$ in boundary condition (in first derivative).

The prove can be obtained immediately when we define the optimization problem of minimizing the error introduced by discrete form of an approximation of a derivative

$$\left| \psi'' - \frac{\psi(x+h) - c_1\psi(x) + \psi(x-h)}{h^2} \right| \tag{4}$$

with introduced $c_1$ constant as a unknown optimization parameter. For the Helmholtz equation that has known solution that may be written as $\psi = Ae^{jkz}$ for which

$\psi''(x) = -k^2\psi(x)$ and $\psi(x \pm h) = e^{\pm jkh}\psi(x)$ the equation could be solved analytically with a following result

$$c_1 = 2\cos(kh) + k^2h^2 \qquad (5)$$

what means that applying $c_1$ as in Eq. (5) guarantees exact solution! The scheme obtained could be expressed in the slightly different form as proposed in Table 1 what proves the lemma.

*Tab. 1. The comparison of standard and nonstandard scheme for Helmholtz one-dimensional equation.*

| | direct discretization form | numerical algorithm form |
|---|---|---|
| standard approach | $\frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{h^2} + k^2\psi_i = 0$ | $\psi_{i+1} - \underbrace{(2 - k^2h^2)}_{\alpha}\psi_i + \psi_{i-1} = 0$ |
| nonstandard approach | $\frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{h^2 \frac{\sin^2\left(\frac{k}{2}h\right)}{\left(\frac{k}{2}h\right)^2}} + k^2\psi_i = 0$ | $\psi_{i+1} - \underbrace{\left(2 - \frac{\sin^2\left(\frac{k}{2}h\right)}{\left(\frac{k}{2}\right)^2}k^2\right)}_{2\cos(kh)}\psi_i + \psi_{i-1} = 0$ |

For the case of two- and three-dimensional Helmholtz PDE the solution is obtained by averaging [2] and the nonstandard approach does not give exact solution, unfortunately. However, the optimised solution have similar form: $c_2 = 4J_0(kh) + k^2h^2$ for two-dimensional PDE and $c_3 = 6j_0(kh) + k^2h^2$ for three-dimensional one where $J_0$ and $j_0$ are Bessel function and spherical Bessel functions.

## Results

To verify quantitatively the theory of nonstandard approach let us consider the model of thickness vibration of piezoelectric circular disc with a radius $a$ and thickness $l$ as presented in Fig.1. Applying driving current $I$ to electrodes located on upper and lower surfaces of a vertically polarized disc results in vibration of it with velocity $u_3$ and $u_2$ , respectively.

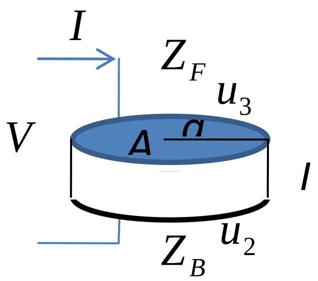


*Fig.1. The parameters of piezoelectric circular disc as used in modelling of impedance frequency response for thickness mode of vibrations.*

The equations presented below describes the analytical and numerical solution required for obtaining input impedance formulae. It will be finally used to compare the performance of standard and nonstandard approach.

The analytical solution could be obtained with application of three-ports model of a disc as described i.e. in [3]. It results in matrix equation :

$$\begin{bmatrix} F_2 \\ F_3 \\ V \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_3}{\tanh(j\omega\tau)} & \frac{Z_3}{\sinh(j\omega\tau)} & \frac{h_{33}}{j\omega} \\ \frac{Z_3}{\sinh(j\omega\tau)} & \frac{Z_3}{\tanh(j\omega\tau)} & \frac{h_{33}}{j\omega} \\ \frac{h_{33}}{j\omega} & \frac{h_{33}}{j\omega} & \frac{l}{j\omega\varepsilon_{33}^S \pi a^2} \end{bmatrix}}_{Z} \begin{bmatrix} u_2 \\ u_3 \\ I \end{bmatrix} \tag{6}$$

where impedance matrix $Z$ depends on angular frequency of vibration $\omega$ and geometrical parameters of a disc ($a$, $l$), physical parameters of a disc material ($\rho, c_{33}, \varepsilon_{33}^S, h_{33}$) and its combinations ($\tau = l\sqrt{\rho / c_{33}}, Z_3 = \pi a^2 \sqrt{\rho c_{33}}$). The wave number r from Helmholtz equation is equal here $k = \omega / v_3 = \omega\sqrt{c_{33}/\rho}$. Than the boundary conditions related to water front loading $Z_F$ and cork backing $Z_B$ as applied to the problem can be inserted in solution as:

$$\begin{bmatrix} u_2 \\ u_3 \\ I \end{bmatrix} = \left( [Z] + \begin{bmatrix} Z_B & 0 & 0 \\ 0 & Z_F & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ V \end{bmatrix} \tag{7}$$

Finally the input impedance could be calculated directly as

$$Z_{in} = \frac{V}{I} \tag{8}$$

Discrete solution could be obtained in turn directly from Helmholtz wave equation and can be presented similarly in matrix form as:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} = \left( \begin{bmatrix} -\alpha & 1 & 0 & 0 & 0 \\ 1 & -\alpha & 1 & 0 & 0 \\ 0 & 1 & -\alpha & 1 & 0 \\ 0 & 0 & 1 & -\alpha & 1 \\ 0 & 0 & 0 & 1 & -\alpha \end{bmatrix} + \begin{bmatrix} -\beta_B & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\beta_F \end{bmatrix} \right)^{-1} \begin{bmatrix} -\gamma \\ 0 \\ 0 \\ 0 \\ \gamma \end{bmatrix} \tag{9}$$

where $\xi_i$ is the amplitude of vibration of $i$-th sample of a space discretized along disc thickness, $\alpha$ is the parameter related to matrix form of partial difference equation and is equal:

$$\alpha = \begin{cases} 2 - (kh)^2 & \text{for standard scheme} \\ \cos(kh) & \text{for nonstandard scheme} \end{cases}$$

and $\beta_{B,F}$ and $\gamma_{B,F}$ is related to boundary condition effects:

$$\beta_{B,F} = \begin{cases} 2jkh\frac{Z_{B,F}}{Z_3} \\ 2j\sin(kh)\frac{Z_{B,F}}{Z_3} \end{cases} \qquad \gamma = \begin{cases} 2jkh\frac{h_{33}}{\omega^2}\frac{I}{Z_3} & \text{for standard scheme} \\ 2j\sin(kh)\frac{h_{33}}{\omega^2}\frac{I}{Z_3} & \text{for nonstandard scheme} \end{cases}$$

with impedances defined as $Z_3 = \pi a^2 \sqrt{\rho c_{33}}$ $Z_{B,F} = \pi a^2 \rho_{B,F} v_{B,F}$. Finally the unknown input impedance could be calculated as:

$$Z_{in} = \frac{l}{j\omega\varepsilon_{33}^S A} - \frac{h_{33}}{I}\left(\xi_n - \xi_1\right) \tag{10}$$

Fig. 2 presents the results obtained by applying Eq. (6,7,8) for analytical solution treated as exact solution and Eq. (9,10) for discrete solution. Moreover the latter is used for standard scheme and for nonstandard scheme. Additionally, two cases for nonstandard scheme are distinguished i.e. standard scheme in boundary condition and nonstandard scheme in boundary condition.

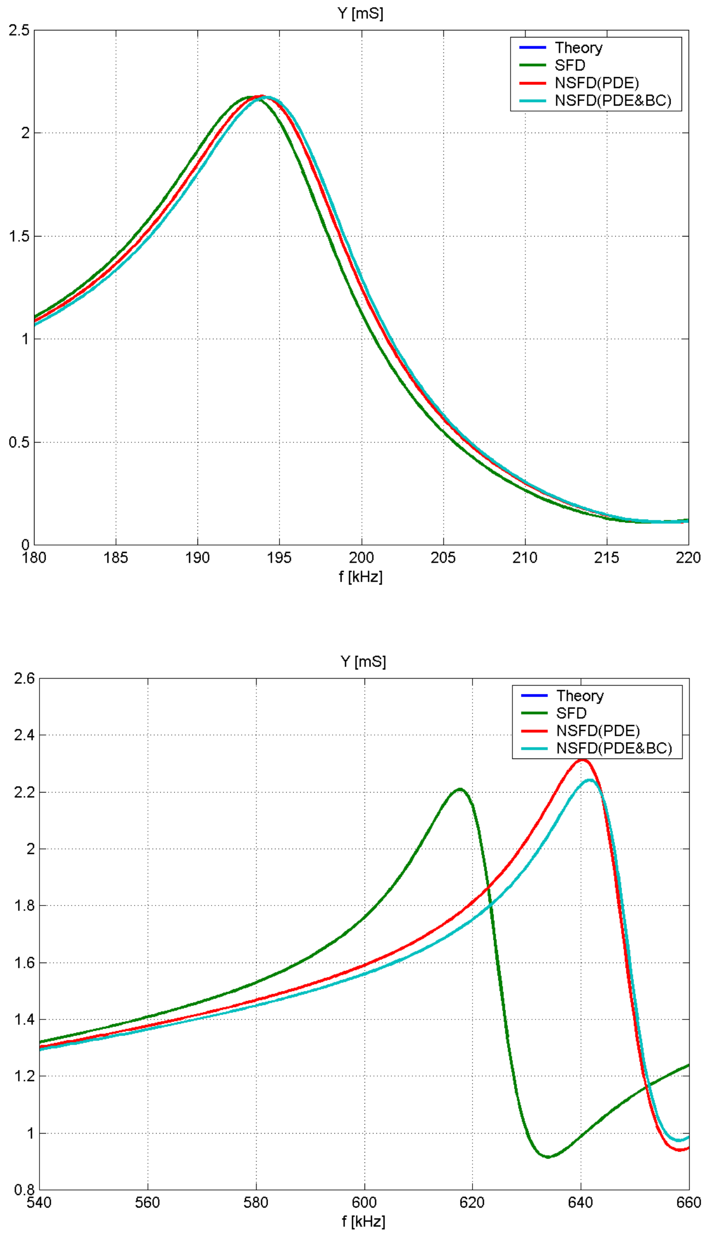


*Fig 2. The admittance frequency response of 200kHz piezoelectric disc in thickness mode of vibration for first (upper chart) and second (lower chart) mode as obtained with analytical and discrete methods. For comparison purposes the discrete solutions are presented for standard scheme, nonstandard scheme applied in PDE only and nonstandard scheme in PDE and in boundary condition.*

As an example let us consider the calculation for one frequency i.e. 200kHz. When a thickness domain $l$=1cm is divided into $n$=11 samples ($kh$=0.29<2π/10 what is below recommended ten wavelengths in disc thickness) the methods gives the modulus of impedance 790Ω, 697Ω and 664Ω, respectively. The last value is exact solution. Obviously, increasing the number if division results in achieving asymptotically the last value, obtained by theory and nonstandard scheme, which is in fact exact in this case. Moreover, it is worth to observe that for higher frequencies the difference in resonance value between standard and nonstandard approach is higher, what confirms known opinion on difficulties in obtaining required accuracy with standard approach for higher vibration modes.

## Discussion

A calculation of admittance for piezoelectric transducer is presented here as an example of numerical solution of the Helmholtz partial differential equation. The results obtained show superior properties of nonstandard approach as compared to standard one. The essential part of nonstandard technique lies in the form of a denominator function used in approximation of derivative and it appears that for Helmholtz wave equation it has a form of cardinal sinus function $\mathrm{sinc}(x)=\sin(x)/x$. It is worth to note that the function is used very often in signal processing in interpolation of band limited signals and is known as a sampling function. Actually the interpolation of signals is related to its spectral representation, which is - in fact - based on orthogonality of Fourier base. But the same Fourier base is also the solution of Helmholtz PDE.

## Conclusions

Nonstandard approach may give exact or optimal solution of differential equation when using discrete numerical methods of solution. It does not require highly precise mesh for discrete calculation. It guarantees stable, consistent and convergent numerical properties of linear operator A of linear equation $\mathbf{A}x=b$ which appears in the process of numerical solution of arbitrary PDE. The approach could be also treated as a specialized regularization of a numerical version of partial difference equation.

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